

Dimensional Deconstruction and Wess–Zumino–Witten Terms

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Abstract

A new technique is developed for the derivation of the Wess-Zumino-Witten terms of gauged chiral lagrangians. We start in $D = 5$ with a pure (mesonless) Yang-Mills theory, which includes relevant gauge field Chern-Simons terms. The theory is then compactified, and the effective $D = 4$ lagrangian is derived using lattice techniques, or “deconstruction,” where pseudoscalar mesons arise from the lattice Wilson links. This yields the WZW term with the correct Witten coefficient by way of a simple heuristic argument. We discover a novel WZW term for singlet currents, that yields the full Goldstone-Wilczek current, and a $U(1)$ axial current for the skyrmion, with the appropriate anomaly structures. A more detailed analysis is presented of the dimensional compactification of Yang-Mills in $D = 5$ into a gauged chiral lagrangian in $D = 4$, heeding the consistency of the $D = 4$ and $D = 5$ Bianchi identities. These dictate a novel covariant derivative structure in the $D = 4$ gauge theory, yielding a field strength modified by the addition of commutators of chiral currents. The Chern-Simons term of the pure $D = 5$ Yang-Mills theory then devolves into the correct form of the Wess-Zumino-Witten term with an index (the analogue of $N_{\text{colors}} = 3$) of $N = D = 5$. The theory also has a Skyrme term with a fixed coefficient.

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I. INTRODUCTION

There is an intriguing parallel between the $D = 5$ pure Yang-Mills theory and the $D = 4$ chiral lagrangian theory of mesons. We first summarize features of the $D = 5$ Yang-Mills theory.

The pure $SU(N)$ Yang-Mills $D = 5$ gauge theory supports a topological soliton, unique to $D = 5$ [1]. This soliton is simply the instanton, an $SU(2)$ configuration, lifted to a time slice in $D = 5$, associated with the nontrivial homotopy class $\Pi_3(SU(2))$, and it carries a conserved topological current [2, 3]. The theory actually has two conserved topological currents, built out of the gauge fields: firstly an adjoint representation current (only for groups possessing d -symbols, e.g., $SU(3)$ and higher), and secondly a singlet current (present for all groups). The adjoint current controls transitions between the various ways in which the instantonic soliton can be embedded into the gauge group (e.g., a pure “I-spin” embedding can flip to a “U-spin” or “V-spin” embedding for $SU(3)$). The singlet current is identically conserved, and yields the topological charge of the soliton.

Each of these currents is topological, and cannot be derived by Noetherian variation of the gauge kinetic term action. The theory must therefore be supplemented with an additional Chern-Simons term. The Chern-Simons term that generates the adjoint current is known as the “second Chern character”, (the $D = 5$ generalization of the Deser-Jackiw-Schonfeld-Siegel-Templeton mass term of $D = 3$ [4, 5]; see also [6]). Under variation of the gauge fields, the second Chern character generates the adjoint current as a source term in the equation of motion of the gauge field. While not manifestly gauge invariant, under small gauge transformations (those continuously connected to the identity), the action containing the second Chern character is invariant. By contrast, for topologically nontrivial transformations, the action shifts by an additive numerical factor, and the coefficient of the Chern character is necessarily quantized so the path integral is invariant (*i.e.*, with the proper coefficient, this shift in the action is then $2\pi N$) [5, 7].

The singlet current has no associated Chern-Simons term built out of the gauge fields alone. We presently propose to introduce a “dual variable,” a vector potential associated with the instantonic soliton. This allows us to write a new $U(1)$ -gauge invariant topological term which is analogous to the second Chern character and which generates the singlet current.

On the other hand, chiral theories of mesons in $D = 4$ based on flavor $SU(N)_L \times SU(N)_R$ also possess remarkable, and quite similar, topological properties. The theories support the skyrmion solution, which is an $SU(2)$ configuration and a stable topological object (whose core is stabilized when the “Skyrme” term is added). The skyrmion also reflects the nontrivial $\Pi_3(SU(2))$, and it carries a conserved (modulo anomalies) singlet current, the Goldstone-Wilczek current [8], which is interpreted as baryon number. The chiral theory also contains adjoint representation topological currents, conserved modulo anomalies. These latter currents again exist only for groups with d -symbols, and govern transitions of the embeddings of the skyrmion in the diagonal subgroup $SU(N)$. A connection between the instantonic soliton of $D = 5$ and the skyrmion of $D = 4$, through compactification, and a matching of their $U(1)$ currents was discussed in some detail in ref.([3]). In fact, the full form of the Goldstone-Wilczek current can be easily inferred from this matching.

The adjoint topological currents in $D = 4$ chiral theories derive from the Wess-Zumino-Witten term. Remarkably, the WZW term is neither manifestly chirally nor gauge invariant; yet it possesses both symmetries for small transformations—those that are continuously connected to the identity. The overall invariance of the path integral under large topological chiral and gauge transformations leads again to quantization conditions on the WZW term coefficient [9]. The singlet Goldstone-Wilczek current has no corresponding WZW term, but as will be discussed below, and elaborated in a companion paper, [10], such a term can be written, provided the σ and η' mesons are incorporated into the theory. This, in turn, leads to a new singlet axial vector topological current.

These topologically interesting aspects of $D = 4$ chiral lagrangians have long been known to follow from the structure of the theory in one higher dimension. In the case of a $D = 4$, $SU(N)_L \times SU(N)_R$ nonlinear σ -model, described by an $N \times N$ unitary matrix field, Witten has shown that the WZW term can be obtained by promoting the global theory of mesons to $D = 5$, where a certain manifestly chirally invariant Chern-Simons term occurs, built out of the mesons. One then compactifies the fifth-dimension term, back to $D = 4$. This results in the global Wess-Zumino term of $D = 4$ [9]. At this stage, by performing gauge transformations upon the global object, one can infer how to introduce the gauge fields to compensate the local changes in the WZ term. This leads to the full Wess-Zumino-Witten term for the gauged chiral lagrangian, which contains the full anomaly structure of the theory. The WZW term plays another crucial role, that of locking the parity of the pion to

the parity of space [9]. Certain allowed transitions, such as $K + \overline{K} \rightarrow 3\pi$, which would be absent without the WZW term, now occur with a topologically quantized amplitude.

A conceptual drawback of this procedure is that local gauge invariance is induced into a non-gauge invariant object, *a posteriori*. Local gauge invariance, however, is a more fundamental symmetry than the chiral symmetry which it breaks, and it is thus preferable to rely on a procedure in which local gauge invariance is present at the outset. Then, upon compactification, one would require that the meson fields appear with their proper kinetic terms and gauging. Implementing such a procedure for compactification, we would expect that the second Chern character of the $D = 5$ pure Yang-Mills theory morphs into the $D = 4$ WZW term. This approach, at least, may shed some light on the interplay on the interrelationship of the various symmetries and topology.

Indeed, there exists in principle such a procedure, the latticization of extra dimensions, [11], or “dimensional deconstruction” [12]. The approach latticizes only the extra dimensions, yielding the effective kinetic and interaction terms, while keeping the $D = 4$ subspace in the continuum. This is related to the earlier “transverse lattice” of Bardeen, Pearson and Rabinovici [13]. For extra dimensional theories, this is a powerful tool, leading to a continuum $D = 4$ effective description of a theory that originated as a pure Yang-Mills theory in higher dimensions, with emphasis on maximal manifest gauge invariance. In this approach, one derives the gauge invariant effective lagrangian for a theory in $D = 4$ that is defined by compactification of a theory in higher dimensions. Starting with pure Yang-Mills in $D = 5$, one can thus engineer a $D = 4$ gauged chiral lagrangian. The mesons then appear in the compactified theory, packaged into exponential chiral fields, which are the Wilson links associated with the latticization.

In the present work, we study the deconstruction of the $D = 5$ Yang-Mills theory, supplemented with the second Chern character, and a new singlet auxiliary term. We begin with a discussion of the physical basis of orbifold boundary conditions, and consideration of the topological aspects of a gauged chiral lagrangian in $D = 4$ through the pure Yang-Mills theory in $D = 5$ without mesons. Presently, we will show how to derive the WZW term for a gauged chiral lagrangian in $D = 4$, by matching of the vector potentials and the field strengths of the $D = 5$ Yang-Mills theory onto the relevant operators in the deconstructed effective lagrangian. We begin in the next section, after a review of $D = 5$ Yang-Mills and the Chern-Simons terms, with a simple heuristic discussion that readily yields the WZW

term on orbifold compactification to $D = 4$. We will also anticipate how a new WZW-like term arises from the singlet auxiliary term in the parent $D = 5$ Yang-Mills theory. This new WZW term generates the singlet Goldstone-Wilczek current, and a new $U(1)$ axial current for the skyrmion.

We then study the general issue of topological deformation of $D = 5$ Yang-Mills into $D = 4$ chiral theories in more detail. First, we note that there is a key element we must address that is missing in a naive deconstruction, and which is essential to the propagation of topology from one geometrical dimension to another. This is the consistency of *the Bianchi identities*. The ordinary $D = 4$ Bianchi identities are always automatically satisfied by the specification that the field strength tensor is a commutator of covariant derivatives, since the $D = 4$ relations are simply the Jacobi identity for antisymmetrized nested commutators. However, we'll find that there is an additional nontrivial Bianchi constraint involving the "lattice hopping derivative." This is seen to fail for the first plausible definition of the hopping derivative in $D = 5$.

We thus formulate the Bianchi identities on the deconstruction lattice, and we find that we are able to satisfy them with a modified definition of the $D = 4$ covariant derivative. The ordinary derivative is modified by the addition of a vector combination of chiral currents with a special coefficient of $1/2$. The formalism automatically implements the "magnetic superconductivity," or confinement phase on the orbifold branes, $G_{4\mu} = 0$.

The Bianchi-consistent modification implies that the effective field strength, $G_{\mu\nu}$, is modified by the addition of terms involving the commutators of chiral currents of the mesons. This term occurs with a fixed coefficient. The gauge action is therefore modified as well, and there now appear in the classical action two Skyrme terms. The usual Skyrme term is generated by the current commutator terms in the field strength tensor and now has a fixed coefficient. Moreover, a new Skyrme term that involves the gauge field, is also present. We thus conjecture that this modified theory may tighten the link between the instantonic soliton in $D = 5$ and the skyrmion in the deconstructed theory in $D = 4$. With these terms, there may exist a "self-dual," and even an analytic skyrmion solution, matching the instantonic soliton at large distances.

Given the new Bianchi-consistent action and field strength, the pure Yang-Mills second Chern character (CS2) term again goes into the WZW term. The resulting WZW term is consistent with Witten's minimal coefficient, but is larger by a factor of 5. Thus, we infer

that the dimensionality of the space-time, $D = 5$, appears as the index of the WZW term in the Bianchi-compliant theory, where we normally would install $N_c = 3$, the number of colors of QCD.

II. THE D=5 PURE YANG-MILLS THEORY AND HEURISTIC DERIVATION OF THE $D = 4$ WESS-ZUMINO-WITTEN TERM

(i) Preliminaries

We start with an $SU(N)$ Yang-Mills gauge theory in $D = 5$. The theory relies on vector potentials, $A_A^a(x)$ and coordinates x^A , where $(A = 0, 1, 2, 3, 4)$, and where x^μ and $(\mu = 0, 1, 2, 3)$ refers to the usual space-time dimensions. When we say “fifth component of a vector, x^A ” we mean, of course, x^4 .

The covariant derivative is

$$D_A = \partial_A - iA_A, \quad A_A = A_A^a Q^a, \quad (1)$$

where Q^a is an abstract operator that takes on the values of $Q^a = \lambda^a/2$ in the adjoint representation. The field strength then is

$$G_{AB} = i[D_A, D_B] = \partial_A A_B - \partial_B A_A - i[A_A, A_B]. \quad (2)$$

This theory has the standard kinetic term:

$$\mathcal{L} = -\frac{1}{2\tilde{g}^2} \text{Tr}(G_{AB}G^{AB}), \quad (3)$$

where $1/\tilde{g}_2^2$ is the coupling with dimensions of mass. With this normalization, gauge fields have the canonical dimensionality with respect to $D = 4$, *i.e.*, $[A_A] = M^1$, and $[G_{AB}] = M^2$.

The theory possesses two identically conserved Chern-Simons currents of the form:

$$J_A = \epsilon_{ABCDE} \text{Tr}(G^{BC}G^{DE}), \quad (4)$$

$$J_A^a = \epsilon_{ABCDE} \text{Tr}\left(\frac{\lambda^a}{2}\{G^{BC}, G^{DE}\}\right). \quad (5)$$

The second current requires that $SU(N)$ possess a d -symbol, hence $N \geq 3$; and it is further *covariantly* conserved,

$$[D^A, J_A^a Q^a] = 0. \quad (6)$$

These topological currents do not arise from eq. (3) under local Noetherian variation of the fields.

The adjoint currents can be derived from an action containing the “second Chern character.” The second Chern character, which we’ll abbreviate as CS2, is derived by ascending to $D = 6$ and considering the generalization of the Pontryagin index (a $D = 6$ generalization of the θ -term),

$$\mathcal{L}_0 = \epsilon_{ABCDEF} \text{Tr} G^{AB} G^{CD} G^{EF}. \quad (7)$$

This can be written as a total divergence,

$$\frac{1}{8} \mathcal{L}_0 = -\partial^F \epsilon_{ABCDEF} \text{Tr} \left(A_A \partial_B A_C \partial_D A_E - \frac{3i}{2} A_A A_B A_C \partial_D A_E - \frac{3}{5} A_A A_B A_C A_D A_E \right). \quad (8)$$

Formally, compactifying the sixth dimension and integrating \mathcal{L}_0 over the boundary in x^5 leads to \mathcal{L}_1 , the second Chern character as an element of the $D = 5$ Lagrangian,

$$\mathcal{L}_1 = c \epsilon^{ABCDE} \text{Tr} \left(A_A \partial_B A_C \partial_D A_E - \frac{3i}{2} A_A A_B A_C \partial_D A_E - \frac{3}{5} A_A A_B A_C A_D A_E \right). \quad (9)$$

This can be rewritten in a convenient form involving gauge covariant field strengths,

$$\mathcal{L}_1 = \frac{c}{4} \epsilon^{ABCDE} \text{Tr} \left(A_A G_{BC} G_{DE} + i A_A A_B A_C G_{DE} - \frac{2}{5} A_A A_B A_C A_D A_E \right), \quad (10)$$

hence, for pure gauge configurations all but the last term vanish. The second Chern characters can be constructed in any odd dimension from a general algorithm [7].

While not manifestly gauge invariant, it is straightforward to verify that CS2 is indeed gauge invariant for gauge transformations continuously connected to the identity. By contrast, for topologically nontrivial gauge transformations, the action shifts by a constant. Hence, the coefficient c must be chosen for effective invariance so that the action shifts by $2\pi N$: the path integral is then invariant. It can be shown that this factor is:

$$c = \frac{1}{48\pi^2}. \quad (11)$$

The variation of the action with respect to the gauge field A^a indeed generates the current of eq.(5) as a source for the equation of motion.

(ii) Heuristic Derivation of $D = 4$ Wess-Zumino-Witten Term

Consider orbifold compactification of the $D = 5$ Yang-Mills theory to a $D = 4$ theory. Orbifold compactification is usually specified mathematically following Horava and Witten [14], such as “compactification on S_1/Z_2 .” One thus considers an interval $0 \leq x^4 \leq 2a$, classifies basis functions as even $P = (+)$ or odd $P = (-)$ under reflection about $x^4 = a$, and under compactification, demands that the $P = (+)$ basis functions are assigned to the $D = 4$ vector potentials, A_μ^A , and the $P = (-)$ basis functions to the A_4^A vector potentials. Orbifolding is the basis of many models of low energy extra dimensions, but we prefer a more physical statement on orbifold compactification.

Alternatively, we can consider two branes to be located at $x^4 = 0$ and $x^4 = a$. Each brane i has a normal vector η_i^A ; *e.g.*, for brane “L” we have $\eta_L = (0, 0, 0, 0, 1)$ and for brane “R” we have $\eta_R = (0, 0, 0, 0, -1)$. The orbifold boundary conditions can be viewed as a special gauge choice for the boundary condition applied on each brane of:

$$\eta_A G^{AB}|_{L,R} = 0. \quad (12)$$

This boundary condition is manifestly gauge invariant. For the η_i defined above, we see that $G_{04} = 0$, hence the normal component of the chromoelectric field strength is zero. Moreover, the “parallel” magnetic field $G_{\mu 4}$ where $\mu \neq 0$ is also zero. These boundary conditions on the branes are dual to those of an electric superconductor, and they thus correspond to a magnetic superconductor. A magnetic superconductor would form electric flux tubes between electric charges (quarks) in the medium, hence a magnetic superconducting phase is a confinement phase.

We thus consider the orbifold compactification as a kind of parallel plate magnetic superconducting capacitor (it can likewise be viewed as a magnetic superconducting Josephson junction). Spanning the gap between the plates, is a Wilson line:

$$U = P \exp \left(-i \int_0^a dx^A A_A \right) = \exp(i\tilde{\pi}), \quad (13)$$

where, upon compactification, we view the Wilson line as a chiral field of mesons, as indicated, with $\tilde{\pi} = \pi^a \lambda^a / f_\pi$, where $f_\pi = 95$ MeV.

In the superconducting boundary brane, or capacitor plate regions (we’ll refer to these generically as the “end-zones”), we can perform local gauge transformations. If the gauge group is $SU(N)$, then there exist gauge transformations V_L (V_R) that are constant over the

entire left-hand (right-hand) end-zone. These can be identified as global $SU(N)_L$ ($SU(N)_R$) transformations. Under these transformations we see that U transforms as

$$U \rightarrow V_L U V_R^\dagger, \quad (14)$$

and the theory under compactification becomes a gauged $SU(N)_L \times SU(N)_R$ chiral lagrangian. The gauge fields should be viewed as left- and right-handed combinations of the normal vector and axial vector mesons of QCD, and they should be supplemented with additional Higgs fields to acquire masses. We thus do not pass to a unitary gauge in which the A_4 modes are eaten by gauge fields to acquire masses.

In the end-zones, we have the magnetic superconducting phase. Here we hypothesize that the vector potentials are determined by “London currents,” the chiral currents built out of the Wilson line:

$$A_A(\text{L end-zone}) = iU[\partial_A, U^\dagger] \equiv i\alpha_A, \quad A_A(\text{R end-zone}) = iU^\dagger[\partial_A, U] \equiv i\beta_A. \quad (15)$$

London currents are generated by the magnetic condensate kinetic term (*e.g.*, analogous to a Higgs field), that locks the vector potential to the Nambu-Goldstone boson, *e.g.*, in our present case $A_A(L) = \partial_A \tilde{\pi} + \dots$ and $A_A(R) = -\partial_A \tilde{\pi} + \dots$ in the endzones. The particular definitions given in eq. (15) are pure gauges, and thus the gauge field strength vanishes (*e.g.*, using form notation, $d\alpha = -\alpha^2$, and $(1/2)G = dA - iA^2 = 0$ when $A = i\alpha$).

We now seek the low energy effective theory. We substitute the London current vector potentials into the Chern-Simons term of eq.(10) to obtain the $D = 4$ effective topological lagrangian:

$$\left(\frac{1}{2 \times 5}\right) \frac{i}{48\pi^2} \epsilon_{ABCDE} (\text{Tr } \alpha^A \alpha^B \alpha^C \alpha^D \alpha^E + \text{Tr } \beta^A \beta^B \beta^C \beta^D \beta^E). \quad (16)$$

where the α (β) terms reside on the left (right) end-zone. To leading order in the expansion in pions, we can write,

$$\begin{aligned} \epsilon_{ABCDE} \text{Tr } \alpha^A \alpha^B \alpha^C \alpha^D \alpha^E &= i\epsilon_{ABCDE} \partial_A \text{Tr } \tilde{\pi} \alpha^B \alpha^C \alpha^D \alpha^E + \dots, \\ \epsilon_{ABCDE} \text{Tr } \beta^A \beta^B \beta^C \beta^D \beta^E &= -i\epsilon_{ABCDE} \partial_A \text{Tr } \tilde{\pi} \beta^B \beta^C \beta^D \beta^E + \dots, \end{aligned} \quad (17)$$

Thus, when we integrate x^4 over the gap between the end-zones, $\int_0^a dx^4$, we arrive at the effective lagrangian,

$$\frac{1}{240\pi^2} \epsilon_{\mu\nu\rho\sigma} (\text{Tr } \tilde{\pi} \alpha^\mu \alpha^\nu \alpha^\rho \alpha^\sigma). \quad (18)$$

Eq.(18) is the precise structure of the leading piece of the Wess-Zumino term in an expansion in pions with Witten's normalization.

A few comments are in order. Note that the expression is hermitian—and it can be written as either $\text{Tr}(\pi\alpha^4)$ or $\text{Tr}(\pi\beta^4)$. Witten's derivation involves compactification on a disk, and the WZW term resides on the periodic boundary of the disk, while the present approach has used the orbifold configuration. Witten writes in an expansion in pions $(2/15\pi^2 F_\pi^5) \text{Tr}(A\partial_\mu A\partial_\nu A\partial_\rho A\partial_\sigma A) + \dots$ with $A = \pi^a \lambda^a$ and $F_\pi = 2f_\pi$, which is consistent with eq.(18). We note that the α terms in the above derivation received a minus sign upon integrating from 0 to a (which canceled against i^2) since the left end-zone resides at the lower limit of the integral; the β terms received a positive sign. In the $D = 4$ theory the currents $\alpha(x_\mu)$ and $\beta(x_\mu)$ are viewed as residing at a common point in $D = 4$ space-time, and we then have identities such as:

$$\epsilon_{ABCDE} \text{Tr} \tilde{\pi} \beta^B \beta^C \beta^D \beta^E = \epsilon_{ABCDE} \text{Tr} U \tilde{\pi} U^\dagger \alpha^B \alpha^C \alpha^D \alpha^E, \quad (19)$$

and $U \tilde{\pi} U^\dagger = \tilde{\pi}$, and we use $U \beta U^\dagger = -\alpha$ and $U \tilde{\pi} U^\dagger$ to bring the two terms into the common form.

With covariant London currents, *e.g.*, $\alpha_A \rightarrow U[D_A, U^\dagger]$, the expression becomes fully gauge invariant. The field strength is then nonzero, and other operators like $\text{Tr}(\pi\alpha^2 G)$, $\text{Tr}(\pi\alpha G\alpha)$, *etc.*, now appear. This expression can be integrated by parts into the full Wess-Zumino-Witten term which will be developed in greater detail elsewhere [10].

The present “derivation” is only meant to be heuristic, and is not well-defined (*e.g.*, operators like $U[D_A, U^\dagger]$ have path dependence). Nonetheless, there are many alternative ways to proceed to formalize the deformation theory from $D = 5$ pure Yang-Mills into $D = 4$ chiral lagrangians. In the subsequent sections we'll be led to a particular and well-defined deformation of the $D = 5$ Yang-Mills theory into a $D = 4$ chiral lagrangian in which, *e.g.*, $A_\mu(L) \rightarrow A_\mu(L) + i\frac{1}{2}U[D_\mu, U^\dagger]$.

(iii) Singlet Auxiliary Chern-Simons Term and a New Singlet WZW Term

We presently turn to the singlet topological current, and we'll merely anticipate some results that follow for the compactification and deconstruction, using the techniques of the next section.

The singlet Chern-Simons current can be generated by an additional modification of the

Lagrangian of the form (CS1):

$$\mathcal{L}_2 = c' \epsilon_{ABCDE} V^A \text{Tr}(G^{BC} G^{DE}), \quad (20)$$

where V^A is a singlet auxiliary vector field. Since it is identically conserved, the CS singlet current couples to this vector field in \mathcal{L}_2 compatibly with a simple abelian gauge-invariance, $\delta V^A = \partial_A \sigma$. If the vector field V is endowed with kinetic terms, the singlet current is also generated as a source in the corresponding Maxwell equations of motion for V . Note that the singlet current cannot be derived from CS2, as the Chern-Simons term of eq.(9) only exists in $SU(N)$ for $N \geq 3$ and does not occur, e.g., in $SU(2)$ Yang-Mills, while the singlet current is always present. One might argue that in $SU(2)$ the form of the current can be inferred, *a posteriori*, e.g., by considering the λ^8 component in $SU(3)$ of the adjoint current, and setting coset fields to zero to descend to $SU(2)$. The singlet current cannot arise from direct variation of CS2, eq.(9), and eq.(20) (CS1) is required to generate it *a priori*.

The appearance of V_A is linked to the instantonic soliton [2, 3], the 't Hooft instanton lifted to a static “monopole” configuration in $D = 5$. This object has a mass of $8\pi^2/\tilde{g}^2$, and it descends to the skyrmion, characterized by the Goldstone-Wilczek current [8], in $D = 4$ [3]. V_A can be interpreted as an effective field associated with the instantonic soliton. The choice of V_A is dictated by the degrees of freedom in the theory. We must generate a conserved current, hence the variation $\delta V^A = \partial_A \sigma$, *i.e.*, we have no complex fields to draw upon. However, the instantonic soliton must be described as a massive excitation, hence we cannot use a Nambu-Goldstone field σ by itself. We may thus infer that the instantonic soliton is associated with a massive $U(1)$ gauge field.

Making use of the chiral deconstruction techniques discussed in the present paper, we can deconstruct eq.(20) to obtain a new auxiliary WZW term that generates the Goldstone-Wilczek current. The field V^A is decomposed into $V_L^A + V_R^B$ where V_L (V_R) has support in the L (R) end-zone. The x^4 integrated zero modes of V^A are then defined in terms of σ and η' fields of a chiral theory of mesons:

$$\begin{aligned} \int dx^4 V_{4R} &= a(\sigma + \eta'), & \int dx^4 V_{4L} &= a(\sigma - \eta'), \\ \int dx^4 V_{\mu R} &= af^{-1} \partial_\mu (\sigma - \eta'), & \int dx^4 V_{\mu L} &= af^{-1} \partial_\mu (\sigma + \eta'), \end{aligned} \quad (21)$$

where the choices are consistent with parities, and the Noether variations that we would make for the original V^A to generate the currents (here we have set the decay constant of

σ and η' to unity). Note that σ and η' can be viewed as glueballs, physical objects in the end-zone phases, even though the theory is quarkless. We find that CS1, using methods developed in the next section, deconstructs to terms containing the following form:

$$\begin{aligned} \mathcal{L}_2 \rightarrow & -\frac{ac'}{2}\eta'\epsilon_{\mu\nu\rho\sigma}\text{Tr}(G_L^{\mu\nu}G_L^{\rho\sigma} + G_R^{\mu\nu}G_R^{\rho\sigma}) + i\frac{ac'}{2f}\partial^\mu(\eta')\epsilon_{\mu\nu\rho\sigma}\text{Tr}(\alpha^\nu G_L^{\rho\sigma} - \alpha^\nu \overline{G}_R^{\rho\sigma}) \\ & -\frac{2ac'}{f}\partial^\mu\sigma\epsilon_{\mu\nu\rho\sigma}\text{Tr}\left(\frac{3i}{2}\alpha^\nu G_L^{\rho\sigma} + \frac{3i}{2}\alpha^\nu \overline{G}_R^{\rho\sigma} + \alpha^\nu\alpha^\rho\alpha^\nu\right) \\ & + \frac{3a}{2}c'\sigma\epsilon_{\mu\nu\rho\sigma}\text{Tr}(G_{L\mu\nu}G_{L\rho\sigma} - G_{R\mu\nu}G_{R\rho\sigma}), \end{aligned} \quad (22)$$

where $\alpha = U[D, U^\dagger]$ and $\beta = U^\dagger[D, U]$, and $\overline{G}_R^{\rho\sigma} = UG_R^{\rho\sigma}U^\dagger$. This is a new WZW-like term that correctly generates the full Goldstone-Wilczek current, [3, 8] with the correct normalization of a unit of baryon number for the skyrmion, provided $c' = 1/48\pi^2$ (identifying $c = c'$ of the second Chern character),

$$Q^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}\left(\alpha_\nu\alpha_\rho\alpha_\sigma + \frac{3i}{2}(G_{L\nu\rho}\alpha_\sigma + \overline{G}_{R\nu\rho}\alpha_\sigma)\right). \quad (23)$$

Thus, by constructing the Noether equation of motion of the σ meson, we generate the full conservation equation of the GW current, including its anomaly,

$$\partial_\mu Q^\mu = -\frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}(G_{L\mu\nu}G_{L\rho\sigma} - G_{R\mu\nu}G_{R\rho\sigma}). \quad (24)$$

This shows that the singlet topological Chern-Simons current matches the full GW current under compactification.

Moreover, by Noether variation of the η' , we obtain a “ $U(1)$ axial current,”

$$Q_5^\mu = \frac{Z}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}(G_{L\nu\rho}\alpha_\sigma - \overline{G}_{R\nu\rho}\alpha_\sigma), \quad (25)$$

This actually has an indeterminate normalization Z . Its divergence equation likewise follows from the η' equation of motion:

$$\partial_\mu Q_5^\mu = \frac{Z}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}[iG_{L\mu\nu}\alpha_\rho\alpha_\sigma + i\overline{G}_{R\mu\nu}\alpha_\rho\alpha_\sigma + G_{L\mu\nu}G_{L\rho\sigma} + G_{R\mu\nu}G_{R\rho\sigma}], \quad (26)$$

and $Z = 1$ is thus favored by matching to the $U(1)$ axial anomaly. The last two terms are the correct form of an axial current anomaly, while the first terms on the *rhs* are analogous to the Skyrme terms. The first two terms form a pseudoscalar and can be interpreted as $2im\overline{\psi}\gamma^5\psi$

in the axial current divergence of a massive nucleon. This current, to our knowledge, has not been previously discussed in the literature. The details of this derivation will be presented elsewhere, [10].

III. DECONSTRUCTION AND BIANCHI IDENTITIES

The heuristic argument presented in section II suggests that a direct morphing of the Chern-Simons terms of $D = 5$ Yang-Mills into $D = 4$ chiral lagrangians is possible and meaningful. We expect that there are many possible deformations of the parent theory in $D = 5$, through deconstruction, that can yield various chiral theories in $D = 4$. These deformations may or may not exploit the full geometrical and topological matching. We would expect that an integral multiple of the minimal coefficient of Witten, $1/240\pi^2$, will always obtain in a consistent theory.

The heuristic argument indeed gave the “minimal coefficient” of the WZ term of Witten. We will now turn to a more literal interpretation of dimensional deconstruction of pure $D = 5$ Yang-Mills which pays closer attention to the details of topological mapping—in particular, to the definition of motion (“hopping”) in the fifth dimension and to the Bianchi identities. Remarkably, the present construction yields the WZW term with a coefficient that is of the form $N/240\pi^2$, where the index $N = D = 5$, is the dimensionality of the parent space-time.

(i) Preliminaries

We presently consider the compactification of the $SU(N)$ Yang-Mills theory in $D = 5$ on the interval $0 \leq x^4 \leq a$. First, construct a coarse-grained lattice of the x^4 dimension with 2 slices. On each slice lies a copy of the gauge group with hermitian generators Q_i^a . The covariant derivative is a sum over all slices with the appropriate abstract charge assigned to each gauge field,

$$D_\mu = \partial_\mu - iA_{L\mu}^a Q_L^a - iA_{R\mu}^a Q_R^a, \quad (27)$$

where we use the notation “left,” L (“right,” R) for brane 1 (2). The generators Q_L and Q_R act on the given slice, and the slices are connected from L to R by a unitary Wilson link U ,

connecting the 1st to the 2nd slice (while the link U^\dagger connects slice 2 to slice 1). Thus,

$$[Q_L, Q_R] = 0. \quad (28)$$

The hermitian field strength tensor is,

$$G_{\mu\nu} = i[D_\mu, D_\nu] = G_{L\mu\nu}^a Q_L^a + G_{R\mu\nu}^a Q_R^a, \quad (29)$$

and also resolves into L and R operator components.

(ii) Matrix Formalism

We choose to define a “left-handed derivative,” $D_{L\mu} = \partial_\mu - iA_{L\mu}^a Q_L^a$, so that $G_{L\mu\nu}^a Q_L^a = i[D_{L\mu}, D_{L\nu}]$; and, respectively, a “right-handed derivative,” $D_{R\mu} = \partial_\mu - iA_{R\mu}^a Q_R^a$, so that $G_{R\mu\nu}^a Q_R^a = i[D_{R\mu}, D_{R\nu}]$ for the right-handed fields. D_L applies to fields on the left-hand lattice slice, while D_R applies on the right-hand slice. We further require $[D_{L\mu}, D_{R\nu}] = 0$, which does not hold, naively; however, we can still implement this construction as a 2×2 matrix representation.

Operators are defined as left-handed and right-handed in the chirality matrix format,

$$\mathcal{O} = \begin{pmatrix} O^L & 0 \\ 0 & O^R \end{pmatrix}. \quad (30)$$

Hence, the matrix covariant derivative can be defined as

$$\mathcal{D}_\mu = \begin{pmatrix} D_\mu^L & 0 \\ 0 & D_\mu^R \end{pmatrix}. \quad (31)$$

The commutator, then, yields the field strengths residing on their respective lattice slices:

$$\mathcal{G}_{\mu\nu} = i[\mathcal{D}_\mu, \mathcal{D}_\nu] = \begin{pmatrix} G_{\mu\nu}^L & 0 \\ 0 & G_{\mu\nu}^R \end{pmatrix}. \quad (32)$$

The gauge transformations in this space are thus,

$$\mathcal{O} \rightarrow \mathcal{V} \mathcal{O} \mathcal{V}^\dagger, \quad \mathcal{V} = \begin{pmatrix} V_L & 0 \\ 0 & V_R \end{pmatrix}. \quad (33)$$

Lattice link fields are off-diagonal matrices,

$$\mathcal{U} = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \mathcal{U}^\dagger = \begin{pmatrix} 0 & 0 \\ U^\dagger & 0 \end{pmatrix}. \quad (34)$$

Note, then,

$$\mathcal{U}^\dagger \mathcal{U} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{U} \mathcal{U}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (35)$$

so that

$$\mathcal{U} \mathcal{U}^\dagger + \mathcal{U}^\dagger \mathcal{U} = \mathbb{1}, \quad \mathcal{U} \mathcal{U}^\dagger - \mathcal{U}^\dagger \mathcal{U} = \sigma_z. \quad (36)$$

The lattice Wilson links transform as bifundamentals,

$$\mathcal{U} \rightarrow \mathcal{V} \mathcal{U} \mathcal{V}^\dagger = \begin{pmatrix} 0 & V_L U V_R^\dagger \\ 0 & 0 \end{pmatrix}, \quad \mathcal{U}^\dagger \rightarrow \mathcal{V} \mathcal{U}^\dagger \mathcal{V}^\dagger = \begin{pmatrix} 0 & 0 \\ V_R U^\dagger V_L^\dagger & 0 \end{pmatrix}. \quad (37)$$

The commutators of operators with link fields are:

$$[\mathcal{O}, \mathcal{U}] = \begin{pmatrix} 0 & O_L U - U O_R \\ 0 & 0 \end{pmatrix}, \quad [\mathcal{O}, \mathcal{U}^\dagger] = \begin{pmatrix} 0 & 0 \\ U^\dagger O_L - O_R U^\dagger & 0 \end{pmatrix}. \quad (38)$$

The abstract charge is defined as

$$\mathcal{Q}^a = \begin{pmatrix} Q_L^a & 0 \\ 0 & Q_R^a \end{pmatrix}. \quad (39)$$

Thus, define the Q^a as having commutators on the U 's:

$$T^a \equiv \frac{\lambda^a}{2}, \quad [Q_L, U] = T^a U, \quad [Q_R, U] = -U T^a. \quad (40)$$

We often encounter these charges sandwiched between U and U^\dagger matrices. We thus see that, e.g.,

$$U^\dagger Q_L^a U = U^\dagger T^a U + Q_L^a. \quad (41)$$

The structure of eq. (38) allows covariant differentiation to be written as a commutation relation, and takes the following form on U ,

$$[D_\mu, U] = \partial_\mu U - i A_{L\mu}^a \frac{\lambda^a}{2} U + i A_{R\mu}^a U \frac{\lambda^a}{2}. \quad (42)$$

This corresponds to the chirality matrix commutator,

$$[\mathcal{D}_\mu, \mathcal{U}] = \begin{pmatrix} 0 & D_{L\mu} U - U D_{R\mu} \\ 0 & 0 \end{pmatrix}. \quad (43)$$

From the link field U , we may thus form left-handed (right-invariant), and right-handed (left-invariant) chiral currents, respectively (non-matrix),

$$\alpha_\mu \equiv U[D_\mu, U^\dagger], \quad \beta_\mu \equiv U^\dagger[D_\mu, U]. \quad (44)$$

More explicitly,

$$\begin{aligned} \alpha_\mu &= U(\partial_\mu - iA_{R\mu}^a \frac{\lambda^a}{2})U^\dagger + iA_{L\mu}^a \frac{\lambda^a}{2} = U(D_{R\mu}U^\dagger - U^\dagger D_{L\mu}), \\ \beta_\mu &= U^\dagger(\partial_\mu - iA_{L\mu}^a \frac{\lambda^a}{2})U + iA_{R\mu}^a \frac{\lambda^a}{2} = U^\dagger(D_{L\mu}U - UD_{R\mu}), \end{aligned} \quad (45)$$

where the action of the derivatives follows Leibniz's rule, $D_L U = [D_L, U] + UD_L$; likewise, $D_R U^\dagger = [D_R, U^\dagger] + U^\dagger D_R$.

In the chiral matrix representation, these amount to

$$\hat{\alpha}_\mu = \mathcal{U}[\mathcal{D}_\mu, \mathcal{U}^\dagger] = \begin{pmatrix} \alpha_\mu & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\beta}_\mu = \mathcal{U}^\dagger[\mathcal{D}_\mu, \mathcal{U}] = \begin{pmatrix} 0 & 0 \\ 0 & \beta_\mu \end{pmatrix}. \quad (46)$$

Finally, it is useful to define the hermitian link chiral matrix:

$$\mathcal{U}_+ \equiv \mathcal{U} + \mathcal{U}^\dagger = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix}. \quad (47)$$

Thus,

$$\mathcal{U}_+ \mathcal{U}_+ = \mathbb{1}, \quad (48)$$

and one sees that

$$\mathcal{A}_\mu \equiv \hat{\alpha}_\mu + \hat{\beta}_\mu = \mathcal{U}[\mathcal{D}_\mu, \mathcal{U}^\dagger] + \mathcal{U}^\dagger[\mathcal{D}_\mu, \mathcal{U}] = \mathcal{U}_+[\mathcal{D}_\mu, \mathcal{U}_+] = \begin{pmatrix} \alpha_\mu & 0 \\ 0 & \beta_\mu \end{pmatrix}. \quad (49)$$

A useful set of relationships that recur throughout, especially in computing current divergences, is

$$\begin{aligned} [D_\mu, \alpha_\nu] - [D_\nu, \alpha_\mu] &= -[\alpha_\mu, \alpha_\nu] - iU[G_{\mu\nu}, U^\dagger], \\ [D_\mu, \beta_\nu] - [D_\nu, \beta_\mu] &= -[\beta_\mu, \beta_\nu] - iU^\dagger[G_{\mu\nu}, U], \end{aligned} \quad (50)$$

with the correspondence in the chirality matrix representation:

$$[\mathcal{D}_\mu, \mathcal{A}_\nu] - [\mathcal{D}_\nu, \mathcal{A}_\mu] = -[\mathcal{A}_\mu, \mathcal{A}_\nu] - i\mathcal{U}_+[\mathcal{G}_{\mu\nu}, \mathcal{U}_+]$$

$$= \begin{pmatrix} iG_{\mu\nu}^L - iUG_{\mu\nu}^R U^\dagger - [\alpha_\mu, \alpha_\nu] & 0 \\ 0 & iG_{\mu\nu}^R - iU^\dagger G_{\mu\nu}^L U - [\beta_\mu, \beta_\nu] \end{pmatrix}. \quad (51)$$

How should \mathcal{D}_4 be defined in D=4? \mathcal{D}_4 is some kind of a lattice derivative, or “brane-hop”, in the x^4 direction. In general, hopping on an N -slice lattice works through the Wilson link, U_i , fields, which are identified with the continuum A^4 through:

$$U_i = P \exp \left(-i \int_{x_i^4}^{x_{i+1}^4} A_4 dx^4 \right). \quad (52)$$

Consider a field $\psi_i(x)$ on the i th slice, where $\psi_i(x) \rightarrow V_i(x)\psi_i(x)$ under local gauge transformations $V_i(x)$ of the local gauge symmetry group on the i th slice. To define a covariant derivative in the x^4 direction, one seeks a difference like $(\psi_{i+1}(x) - \psi_i(x))/a$, for lattice spacing a . But since this has a mixed gauge symmetry, one is led to define the covariant difference $(U_i\psi_{i+1}(x) - \psi_i(x))/a$. The link now pulls the first term back from the slice $i+1$ to i , where the i -covariant difference can be computed, invariant under $i+1$ transformations. A vanishing covariant difference thus amounts to link-gauge transformation. For adjoint quantities, both sides of the corresponding operator need such adjustment.

The hopping derivative in deconstruction must handle left and right in a manner consistent with parity. One possibility would be to define an *off diagonal* (antihermitian) hopping derivative as a commutator, and thus traceless,

$$[\mathcal{D}^4, \mathcal{O}] \equiv -\frac{1}{a}[\mathcal{U}_+, \mathcal{O}] = -\frac{1}{a} \begin{pmatrix} 0 & -UO_R + O_L U \\ O_R U^\dagger - U^\dagger O_L & 0 \end{pmatrix}. \quad (53)$$

With this definition, the hopping derivative obeys Leibniz’s chain rule of differentiation, as a commutator, and so the Bianchi identities in $D = 4$ are automatically satisfied, so there is no need for modification of the theory. Thus, an orbifold compactification solves the Bianchi identity with the usual spectrum. In addition, by Leibniz’s rule, $\mathcal{O}\psi$ hop-transforms exactly like ψ . This may be at the root of a deficiency, however. Being off diagonal, this \mathcal{D}_4 maps operators from one representation into another. For example, it maps an adjoint representation under $SU(N)_L$, *i.e.*, $(N^2 - 1, 0)$, into a bifundamental under $SU(N)_L \times SU(N)_R$, *i.e.*, (N, \overline{N}) . A covariant derivative then which does not faithfully map a given representation into itself is unsatisfactory.

Moreover, if it is applied to fermions, one immediately encounters the fermion doubling problem. The remedy to this is the addition of a Wilson term, which is a continuum second

derivative. If we generalize the Wilson term to the case of higher representations, such as adjoint operators, we are led to the diagonal definition, below [16]. The Wilson term projects out the unwanted fermionic doublers, and permits the appearance of anomalies consistently with topology. Multiplication of the above \mathcal{D}_4 by $-\sigma_z \mathcal{U}_+$ on the left, however, leads to a different, *diagonal hopping derivative* defined below.

A preferred definition, and the one we will be using presently is a *diagonal hopping derivative*,

$$\mathcal{D}^4(\mathcal{O}) \equiv \frac{1}{a} (\mathcal{U}[\mathcal{O}, \mathcal{U}^\dagger] - \mathcal{U}^\dagger[\mathcal{O}, \mathcal{U}]) = \begin{pmatrix} U O_R U^\dagger - O_L & 0 \\ 0 & O_R - U^\dagger O_L U \end{pmatrix}, \quad (54)$$

where a is the spacing between neighboring slices.

Note the second term pushes the previous slice fields forward, as the first term pulls the subsequent slice fields back, hence a relative sign difference, which is commensurate with parity: Under parity, $L \leftrightarrow R$, $U \leftrightarrow U^\dagger$, and $\mathcal{D}^4 \rightarrow -\mathcal{D}^4$, and hence the definitions are parity invariant. It is important to note, however, that, like lattice derivatives, this derivative does not obey the Leibniz rule of differentiation, and so *cannot* be written as a commutator $[\mathcal{D}^4, \mathcal{A}\mathcal{B}] = [\mathcal{D}^4, \mathcal{A}]\mathcal{B} + \mathcal{A}[\mathcal{D}^4, \mathcal{B}]$.

We thus define the coset field strength as a transform, not a commutator, through the diagonal hopping derivative:

$$\begin{aligned} \mathcal{G}_{4\mu} &= -\mathcal{G}_{\mu 4} \equiv i\mathcal{D}_4(\mathcal{D}_\mu) \\ &= \frac{i}{a} (\mathcal{U}[\mathcal{D}_\mu, \mathcal{U}^\dagger] - \mathcal{U}^\dagger[\mathcal{D}_\mu, \mathcal{U}]) = \frac{i}{a} (\hat{\alpha}_\mu - \hat{\beta}_\mu) = \frac{i}{a} \begin{pmatrix} \alpha_\mu & 0 \\ 0 & -\beta_\mu \end{pmatrix}. \end{aligned} \quad (55)$$

The conventional deconstructed lagrangian in the chirality matrix formalism can then be written,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2g^2} (\text{Tr } \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} - \text{Tr } \mathcal{G}_{4\nu} \mathcal{G}^{4\nu}) \\ &= -\frac{1}{2g^2} \text{Tr } G_{L\mu\nu} G^{L\mu\nu} - \frac{1}{2g^2} \text{Tr } G_{R\mu\nu} G^{R\mu\nu} - \frac{1}{8} f_\pi^2 \left(\text{Tr}(\alpha_\mu)^2 + \text{Tr}(\beta_\mu)^2 \right), \end{aligned} \quad (56)$$

where we identify $1/g^2 = a/\tilde{g}^2$, and $f_\pi^2 = 4/a\tilde{g}^2 = 1/a^2 g^2$.

It could be interpreted as a gauged chiral lagrangian with external vector fields, A_L^μ and A_R^μ . We may wish to assign the octet of vector mesons, including the ρ to a vector combination of the fields, and the axial vector mesons to the axial vector combination. To

do this in detail would require additional Higgs fields to give masses to the vector (ρ) and axial vector (A_1) combinations. Once these combinations have acquired with longitudinal degrees of freedom, then one cannot eliminate the mesons by gauge transformations.

As an effective fundamental theory, this represents a massless zero mode together with a massive KK mode. To see this, pass to unitary gauge to remove the spinless mesons altogether, i.e., note that $\text{Tr}(\alpha_\mu)^2 = \text{Tr}(\beta_\mu)^2$, and introduce a ‘‘Stückelberg’’ field,

$$V_\mu \equiv -i\alpha_\mu/g. \quad (57)$$

The corresponding field strength, by eq. (50) is:

$$\begin{aligned} F_{\mu\nu}^V &= [D_\mu, V_\nu] - [D_\nu, V_\mu] - i[V_\mu, V_\nu] \\ &= -\frac{1}{g}U[G_{\mu\nu}, U^\dagger] = \frac{1}{g}G_{L\mu\nu}^a \frac{\lambda^a}{2} - \frac{1}{g}UG_{R\mu\nu}^a \frac{\lambda^a}{2}U^\dagger. \end{aligned} \quad (58)$$

Further, the orthogonal zero-mode field strength is likewise right-invariant,

$$F_{\mu\nu}^0 = \frac{1}{g} \left(UG_{R\mu\nu}^a \frac{\lambda^a}{2}U^\dagger + G_{L\mu\nu}^a \frac{\lambda^a}{2} \right). \quad (59)$$

Thus, the effective lagrangian takes the form,

$$\mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{1}{2} \text{Tr} F_{\mu\nu}^V F^{V\mu\nu} - \frac{1}{4} g^2 f_\pi^2 \text{Tr} V_\mu V_\mu, \quad (60)$$

describing a massless zero mode and massive KK mode of mass $gf_\pi/\sqrt{2}$. (The spinless mesons have been absorbed into the longitudinal components of V_μ .)

Note that one can always perform a left gauge transformation on these fields, $D_R \rightarrow UD_RU^\dagger = D'_R$ leading to $G_{R\mu\nu}^a{}' = UG_{R\mu\nu}^a \frac{\lambda^a}{2}U^\dagger$, hence $gF_{\mu\nu}^V = G_{L\mu\nu}^a \frac{\lambda^a}{2} - G_{R\mu\nu}^a{}' \frac{\lambda^a}{2}$; thus $gF_{\mu\nu}^0 = G_{R\mu\nu}^a{}' \frac{\lambda^a}{2} + G_{L\mu\nu}^a \frac{\lambda^a}{2}$. With these field redefinitions, evidently only one linearly realized symmetry transforms all fields, the vectorial symmetry, $O \rightarrow VOV^\dagger$, where $V = V_L$.

(iii) Bianchi Identities

The *Bianchi identities* in $D = 5$ are just the Jacobi identities for covariant derivatives,

$$\epsilon_{ABCDE}[D^C, G^{DE}] = i\epsilon_{ABCDE}[D^C, [D^D, D^E]] = 0. \quad (61)$$

Consistency in $D = 4$ requires:

$$\epsilon_{\mu\nu\rho\sigma}[\mathcal{D}^\nu, \mathcal{G}^{\rho\sigma}] = i\epsilon_{\mu\nu\rho\sigma}[\mathcal{D}^\nu, [\mathcal{D}^\rho, \mathcal{D}^\sigma]] = 0, \quad (62)$$

as well as,

$$\epsilon_{\mu\nu\rho\sigma}\mathcal{D}^4(\mathcal{G}^{\mu\nu}) = \epsilon_{\mu\nu\rho\sigma} ([\mathcal{D}^\mu, \mathcal{G}^{4\nu}] - [\mathcal{D}^\nu, \mathcal{G}^{4\mu}]) . \quad (63)$$

Eq. (62) holds automatically in the $D = 4$ theory, as $\mathcal{G}_{\mu\nu}$ is defined as a commutator of covariant derivatives, for *any choice of* \mathcal{D}_μ .

The off-diagonal hopping derivative satisfies the coset identity eq. (63), while the diagonal hopping derivative, not being a commutator, does not, in general: the Bianchi relation implies a nontrivial constraint. Consider the diagonal hopping on the *lhs* of eq.(63),

$$\mathcal{D}^4(\mathcal{G}^{\mu\nu}) = \frac{1}{a} (\mathcal{U}[\mathcal{G}_{\mu\nu}, \mathcal{U}^\dagger] - \mathcal{U}^\dagger[\mathcal{G}_{\mu\nu}, \mathcal{U}]) , \quad (64)$$

and compare to the *rhs* of eq.(63),

$$\begin{aligned} i[\mathcal{D}_\mu, \mathcal{D}_4(\mathcal{D}_\nu)] - i[\mathcal{D}_\nu, \mathcal{D}_4(\mathcal{D}_\mu)] &= \frac{i}{a} ([\mathcal{D}_\mu, \mathcal{U}[\mathcal{D}_\nu, \mathcal{U}^\dagger]] - [\mathcal{D}_\mu, \mathcal{U}^\dagger[\mathcal{D}_\nu, \mathcal{U}]] - (\mu \leftrightarrow \nu)) \\ &= \frac{i}{a} (\mathcal{U}[[\mathcal{D}_\mu, \mathcal{D}_\nu], \mathcal{U}^\dagger] - \mathcal{U}^\dagger[[\mathcal{D}_\mu, \mathcal{D}_\nu], \mathcal{U}] \\ &\quad + [\mathcal{D}_{[\mu}, \mathcal{U}][\mathcal{D}_{\nu]}, \mathcal{U}^\dagger] - [\mathcal{D}_{[\mu}, \mathcal{U}^\dagger][\mathcal{D}_{\nu]}, \mathcal{U}]) \\ &= \mathcal{D}_4(\mathcal{G}^{\mu\nu}) + \frac{i}{a} \left(-[\hat{\alpha}_\mu, \hat{\alpha}_\nu] + [\hat{\beta}_\mu, \hat{\beta}_\nu] \right) . \end{aligned} \quad (65)$$

The first term of the *rhs* is consistent, but the last term is an unwanted nonvanishing current commutator. This term is nonzero, and is the current algebra of the chiral theory. Thus, the Bianchi identity fails given the presence of this term.

Nonetheless, the constraint, eq.(63), can be satisfied if we consider a *modified covariant derivative*. We find that the desired modification takes the form,

$$\boxed{\mathcal{D}'_\mu \equiv \mathcal{D}_\mu + \frac{1}{2}\mathcal{A}_\mu} . \quad (66)$$

The Bianchi identities of eq.(62) thus remain automatic in the $D = 4$ subspace, since the gauge field strengths are defined, as usual, by commutators of \mathcal{D}'_μ . The Bianchi constraint, eq. (63), now requires the vanishing of the following expression, with the modified derivative:

$$\epsilon^{\mu\nu\rho\sigma} ([\mathcal{D}'_\mu, \mathcal{U}][\mathcal{D}'_\nu, \mathcal{U}^\dagger] - [\mathcal{D}'_\mu, \mathcal{U}^\dagger][\mathcal{D}'_\nu, \mathcal{U}]) = 0. \quad (67)$$

To see the vanishing of this constraint, we first note:

$$\mathcal{U}_+[\mathcal{A}_\mu, \mathcal{U}_+] = -2\mathcal{A}_\mu, \quad (68)$$

hence,

$$\{\mathcal{U}_+, \mathcal{A}_\mu\} = 0, \quad (69)$$

and so, by eqn. (49),

$$[\mathcal{D}'_\mu, \mathcal{U}_+] = 0. \quad (70)$$

It is evident that this, in fact, resolves into the two components,

$$[\mathcal{D}'_\mu, \mathcal{U}] = 0, \quad [\mathcal{D}'_\mu, \mathcal{U}^\dagger] = 0, \quad (71)$$

and the Bianchi constraint is therefore satisfied. We remark that one can derive the same result without recourse to the matrix representation by careful analysis, where, allowing an arbitrary factor w in the current part of eq.(66), one obtains the unwanted current commutators of eq.(65), multiplied by a factor of $(1 - 4w + 4w^2)$. The Bianchi constraint is thus satisfied with the new covariant derivative of eq.(66) with the special coefficient of $w = 1/2$. The matrix formulation both streamlines and automates this derivation.

Observe that the field strength $\mathcal{G}'_{4\mu}$ of (55) manifestly vanishes for this hopping-flat modified derivative,

$$\mathcal{G}'_{4\mu} = i\mathcal{D}_4(\mathcal{D}'_\mu) = 0. \quad (72)$$

(Actually, by $\mathcal{D}_4(\mathcal{G}'_{\mu\nu}) = 0$, each of the three terms in the respective coset identity eq. (63) vanishes separately for modified covariant derivatives.)

The rest of the field strength tensor, by (51), reduces to

$$\mathcal{G}'_{\mu\nu} = i[\mathcal{D}'_\mu, \mathcal{D}'_\nu] = \frac{1}{2}(\mathcal{G}_{\mu\nu} + \mathcal{U}_+ \mathcal{G}_{\mu\nu} \mathcal{U}_+ - \frac{i}{2}[\mathcal{A}_\mu, \mathcal{A}_\nu]), \quad (73)$$

so that

$$\mathcal{G}'_{\mu\nu} = \frac{1}{2} \begin{pmatrix} G_{\mu\nu}^L + U G_{\mu\nu}^R U^\dagger - \frac{i}{2}[\alpha_\mu, \alpha_\nu] & 0 \\ 0 & G_{\mu\nu}^R + U^\dagger G_{\mu\nu}^L U - \frac{i}{2}[\beta_\mu, \beta_\nu] \end{pmatrix}. \quad (74)$$

Evidently, the right-slice amounts to the gauge-transformed image theory of the left-slice,

$$\mathcal{G}'_{\mu\nu} = \frac{1}{2} \begin{pmatrix} G_{\mu\nu}^L + U G_{\mu\nu}^R U^\dagger - \frac{i}{2}[\alpha_\mu, \alpha_\nu] & 0 \\ 0 & U^\dagger (U G_{\mu\nu}^R U^\dagger + G_{\mu\nu}^L - \frac{i}{2}[\alpha_\mu, \alpha_\nu]) U \end{pmatrix}, \quad (75)$$

For gauge-invariant combinations, this ‘‘hop-invariant’’ setup effectively doubles up the theory. The effective field strength appearing here is simply the (hop-symmetric) zero-mode combination encountered previously,

$$F_{\mu\nu}^0 = G_{\mu\nu}^L + U G_{\mu\nu}^R U^\dagger, \quad (76)$$

whereas the orthogonal KK-mode combination is absent.

In effect, the diagonal hopping derivative Bianchi-compatible theory of the two-slice orbifold contains only one propagating gauge field, together with the spinless mesons. This does not mean that there is no KK mode, but that the simplest hop-symmetric deconstruction truncates the spectrum on the propagating zero mode. To obtain the second KK mode would require that we start with $N = 3$ branes, and we would expect that, for any N , the Bianchi-improved theory would describe the zero mode and $N - 2$ KK modes.

Note that the chiral feature of an orbifold is still present, *i.e.*, we may treat $F_{\mu\nu}^0$ as any combination of left-hand or right-hand gauging. For example, we may gauge only the left-hand side of the meson fields, whence setting $G_{\mu\nu}^R = 0$, so that $F_{\mu\nu}^0 = G_{\mu\nu}^L$; or, else, we may choose to gauge isospin, $G_{\mu\nu}^L = G_{\mu\nu}^R$, so $F_{\mu\nu}^0 = 2G_{\mu\nu}^L$ (which rescales the coupling constant).

In the simplifying case that we set the right-hand Yang-Mills fields to zero (*i.e.*, we retain only a single $SU(N)_L$ gauge group), we end up with a pure left-handed chiral theory:

$$\mathcal{G}'_{\mu\nu} = \frac{1}{2} \begin{pmatrix} G_{\mu\nu}^L - \frac{i}{2}[\alpha_\mu, \alpha_\nu] & 0 \\ 0 & U^\dagger (G_{\mu\nu}^L - \frac{i}{2}[\alpha_\mu, \alpha_\nu]) U \end{pmatrix}. \quad (77)$$

The resulting gauge action is then,

$$-\frac{1}{2\tilde{g}^2} \text{Tr} \mathcal{G}'_{\mu\nu} \mathcal{G}'^{\mu\nu} = -\frac{1}{4\tilde{g}^2} \left(\text{Tr} G_{\mu\nu}^L G^{L\mu\nu} - i \text{Tr}(G^{L\mu\nu}[\alpha_\mu, \alpha_\nu]) - \frac{1}{4} \text{Tr}[\alpha_\mu, \alpha_\nu][\alpha^\mu, \alpha^\nu] \right). \quad (78)$$

The resulting theory has several interesting properties evident at this point. The last term, $\text{Tr}([\alpha, \alpha]^2)$, is the Skyrme term required for the stability of the core of the Skyrmion solution. It is normally a puzzle to understand how these terms are generated in a deconstructed theory, since they are needed classically, because the skyrmion core is not an entirely short-distance structure. The Bianchi identities have fixed the coefficient of the Skyrme terms to definite values. While one could always add other contributions to the Skyrme terms by hand, their appearance here reflects self-consistency with the parent $D = 5$ theory, which admits stable large instantonic solitons, which, in turn, carry the current that matches to the Skyrmionic current.

We note that the new cross-term of the form $G^L[\alpha, \alpha]$, which is allowed by the presence of the gauge field. This term has significant effects upon the mass of the skyrmion, and bounds related to those of magnetic monopoles arise [17].

We are thus led to speculate that this Bianchi-consistent theory, with these fixed Skyrme terms, points to a more intricate relationship between the instantonic soliton and the

skyrmion. Perhaps we could now find a skyrmion solution that is “self-dual,” matching the self-duality of the instantonic soliton in $D = 5$, which, in turn, is a consequence of the self-duality of the instanton.

In non-matrix notation, the modified derivative reads,

$$D'_\mu = \partial_\mu - i(A_{L\mu} + \frac{i}{2}\alpha_\mu) \cdot Q_L - i(A_{R\mu} + \frac{i}{2}\beta_\mu) \cdot Q_R, \quad (79)$$

and hence,

$$D'_{L\mu} = \frac{1}{2}(D_{L\mu} + U D_{R\mu} U^\dagger) = \partial_\mu - iA_{\mu L} + \frac{1}{2}\alpha_\mu, \quad D'_{R\mu} = \frac{1}{2}(D_{R\mu} + U^\dagger D_{L\mu} U) = \partial_\mu - iA_{\mu R} + \frac{1}{2}\beta_\mu. \quad (80)$$

Effectively, the gauge fields are augmented by the meson currents α_μ and β_μ . In the limit of vanishing gauge fields, the effective primed gauge fields are still non-trivial,

$$\boxed{A'_{\mu L} \rightarrow \frac{i}{2}U\partial_\mu U^\dagger, \quad A'_{\mu R} \rightarrow \frac{i}{2}U^\dagger\partial_\mu U}, \quad (81)$$

reminiscent of the London equation inside a superconducting medium. Since they are not pure gauges, because of the coefficient of $1/2$, they yield nonvanishing primed field strengths, and hence the Skyrme term exhibited above.

To summarize, the deconstruction prescription we have been led to is based on the diagonal hopping derivative \mathcal{D}_4 ; the Bianchi-consistent hopping-flat modified covariant derivatives, \mathcal{D}'_μ ; and the corresponding field strengths, $\mathcal{G}'_{\mu\nu}$. Having rejected the nonvanishing $\mathcal{G}_{\mu 4}$, in favor of its vanishing primed counterpart, we have forfeited the meson currents' kinetic term, in the naive chiral lagrangian above. To recover them, we might, for instance, supplement the lagrangian with a term of the form:

$$\sim \frac{f_\pi^2}{8} \text{Tr } \mathcal{A}_\mu \mathcal{A}_\mu, \quad (82)$$

or somehow match $\mathcal{A}_\mu \mapsto \mathcal{G}_{4\mu}$. This is equivalent to defining $\mathcal{G}_{4\mu}$ as an off-diagonal operator using the off-diagonal hopping derivative. Another possibility, more consistent with Wilson fermions, is a hybrid hopping derivative that is a combination of the off-diagonal Leibnitz form and the diagonal form discussed above (see [16]; this happens automatically with supersymmetric deconstruction in which hopping terms are defined as superpotentials). We will never need this operator in the derivation of the usual WZW term in the subsequent section, so these ambiguities are irrelevant. We will need the fact, however, that the diagonal $\mathcal{G}'_{4\mu} = 0$.

Ultimately, such prescriptions codify a number of implicit choices of brane configurations and phenomenological outcomes. Unlike the off-diagonal antihermitian hopping derivative, the hermitian diagonal one preserves topological structures associated with chirality (*e.g.*, anomalies).

IV. DERIVATION OF THE WZW TERM IN THE BIANCHI THEORY

The CS2 lagrangian may be written in a form more suitable for subsequent considerations. Specifically, we start by separating the A_4 component,

$$\begin{aligned}\mathcal{L}_1 &= \mathcal{L}_{1a} + \mathcal{L}_{1b}, \\ \mathcal{L}_{1a} &= \frac{c}{4}\epsilon^{\mu\nu\rho\sigma} \text{Tr}(A_4 G_{\mu\nu} G_{\rho\sigma} + iA_4 A_\mu A_\nu G_{\rho\sigma} + iA_4 A_\mu G_{\nu\rho} A_\sigma + iA_4 A_\mu G_{\nu\rho} A_\sigma \\ &\quad - 2A_4 A_\mu A_\nu A_\rho A_\sigma), \\ \mathcal{L}_{1b} &= \frac{c}{2}\epsilon^{\mu\nu\rho\sigma} \text{Tr}(A_\mu G_{\nu\rho} G_{\sigma 4} + A_\mu G_{\nu 4} G_{\rho\sigma} + iA_\mu A_\nu A_\rho G_{\sigma 4}).\end{aligned}\tag{83}$$

This helps re-express \mathcal{L}_{1a} as a lower CS covariant current divergence plus an anomaly term,

$$\mathcal{L}_{1a} = -\frac{c}{2} \text{Tr}(A_4 [D_\mu, K^\mu]) + \frac{3c}{4}\epsilon^{\mu\nu\rho\sigma} \text{Tr}(A_4 G_{\mu\nu} G_{\rho\sigma}),\tag{84}$$

where

$$K^\mu \equiv \epsilon^{\mu\nu\rho\sigma} (iA_\nu A_\rho A_\sigma + G_{\nu\rho} A_\sigma + A_\nu G_{\rho\sigma}).\tag{85}$$

Likewise, since $G_{\mu 4} = [D_\mu, A_4] - \partial_4 A_\mu$, the second term can be written as

$$\mathcal{L}_{1b} = -\frac{c}{2} \text{Tr}([D_\mu, A_4] - \partial_4 A_\mu) K^\mu.\tag{86}$$

The combined CS2 is then

$$\mathcal{L}_1 = \frac{c}{2} \text{Tr}((\partial_4 A_\mu) K^\mu) + \frac{3c}{4}\epsilon^{\mu\nu\rho\sigma} \text{Tr}(A_4 G_{\mu\nu} G_{\rho\sigma}),\tag{87}$$

where some total divergences have been discarded.

Our problem is the interpretation of the first term above. This problem is obviated when $G_{\mu 4} = 0$, whence we use eq.(92) for the full lagrangian. We then need to interpret $[D_\mu, A_4]$. Consider the definition of the Wilson line, which we identify with the chiral field of mesons:

$$U = \exp(-i \int A_4 dx^4) = \exp(i\tilde{\pi}),\tag{88}$$

where, for a zero-mode A_4 we can neglect path-ordering. We can then write, upon expanding the U 's to second order (this is the order necessary for consistent WZW terms—see below):

$$\alpha_\mu = U[D_\mu, U^\dagger] = -i[D_\mu, \tilde{\pi}] - \frac{1}{2}(\tilde{\pi}[D_\mu, \tilde{\pi}] - [D_\mu, \tilde{\pi}]\tilde{\pi}) + O(\tilde{\pi}^3), \quad (89)$$

$$\alpha_\mu = U[D_\mu, U^\dagger] = -i[D_\mu, \int dx^4 A_4] - \frac{1}{2}(\tilde{\pi}[D_\mu, \int dx^4 A_4] - [D_\mu, \int dx^4 A_4]\tilde{\pi}) + \dots \quad (90)$$

We invert this to make the identification

$$[D_\mu, \int dx^4 A_4] = i\alpha_\mu - \frac{1}{2}(\tilde{\pi}\alpha_\mu - \alpha_\mu\tilde{\pi}) + \dots \quad (91)$$

We now impose the condition that, from our Bianchi-improved theory, $G_{4\mu} = 0$, equivalently, $\partial_4 A_\mu = [D_\mu, A_4]$, and we substitute eq.(91) into the expression eq.(87). The full lagrangian upon integrating over x^4 becomes,

$$\mathcal{L}_1 = i\frac{c}{2}\text{Tr}(\alpha_\mu K^\mu) - \frac{c}{4}\text{Tr}(\tilde{\pi}\alpha_\mu K^\mu - \tilde{\pi}K^\mu\alpha_\mu) + \frac{3c}{4}\epsilon^{\mu\nu\rho\sigma}\text{Tr}(\tilde{\pi}G_{\mu\nu}G_{\rho\sigma}) + \dots, \quad (92)$$

We can now check that we recover the Wess-Zumino term. Turn off the gauge fields, but make the deconstructive replacement, with the modified vector potential and field strength summarized in (81), with the primes omitted,

$$A_\mu \rightarrow i\frac{\alpha_\mu}{2}, \quad \text{hence,} \quad G_{\mu\nu} \rightarrow -\frac{i}{2}[\alpha_\mu, \alpha_\nu], \quad K_\mu \rightarrow \frac{5}{8}\epsilon_{\mu\nu\rho\sigma}\alpha^\nu\alpha^\rho\alpha^\sigma. \quad (93)$$

Owing to the vector potential which is no longer a pure gauge (due to the factor of $1/2$), the $G_{\mu\nu}$ terms are now non-negligible and active in our expression for the second Chern character, and this modifies the WZW term's overall coefficient from the heuristic argument result in which $G_{\mu\nu} = 0$. The CS term thus becomes on the left end-zone (the (11) matrix element contribution to the trace):

$$\mathcal{L}_{1L} = -\frac{c}{2}\epsilon_{\mu\nu\rho\sigma}\text{Tr}(\tilde{\pi}\alpha^\mu\alpha^\nu\alpha^\rho\alpha^\sigma) + \dots \quad (94)$$

From the right end-zone, we likewise get the result:

$$\mathcal{L}_{1R} = -\frac{c}{2}\epsilon_{\mu\nu\rho\sigma}\text{Tr}(U^\dagger\tilde{\pi}U\beta^\mu\beta^\nu\beta^\rho\beta^\sigma) + \dots, \quad (95)$$

which is equivalent, since $\tilde{\pi} = U\tilde{\pi}U^\dagger$.

Thus, combining, we obtain the Wess-Zumino term for the Bianchi-consistent theory:

$$\mathcal{L}_1 = -\frac{N}{240\pi^2}\epsilon_{\mu\nu\rho\sigma}\text{Tr}(\tilde{\pi}\alpha^\mu\alpha^\nu\alpha^\rho\alpha^\sigma) + \dots, \quad (96)$$

where the “index” N is given by the dimensionality of the parent theory space-time:

$$N = D = 5. \quad (97)$$

Evidently, the “t Hooft matching” of our Bianchi improved theory intrinsically identifies $D = 5$, reflected in the value of this index. We have no deeper interpretation for this result at present.

Parenthetically, we may suggest care in manipulating the WZ term. For example, we could write (using forms, and $d\alpha = -\alpha^2$ when the vector potential is ignored):

$$\text{tr}(\tilde{\pi}\alpha^4) = \text{tr}(\tilde{\pi}d\alpha d\alpha) = \text{tr}(d\tilde{\pi}\alpha^3). \quad (98)$$

By naively replacing $d\tilde{\pi} \rightarrow i\alpha$, we get zero for the *rhs* by cyclicity of the ϵ -symbol, $\text{Tr}(\alpha^4) = 0$; so we would only access the vanishing, leading part of the WZ term: zero! Of course, at the next order in the expansion in pions, we recover the properly modified covariant derivative,

$$d\tilde{\pi} \rightarrow i\alpha - (1/2)[\tilde{\pi}, \alpha], \quad (99)$$

and hence consistency for the WZ term, to leading non-trivial order.

The higher orders for the WZ term have been discussed mathematically in, *e.g.*, [18]. Beyond leading order, however, the WZ term is not universal in form, as an expansion in pions. Indeed, the expansion of unitary chiral fields, such as $U = \exp(i\tilde{\pi})$, as a power series in $\tilde{\pi}$ is non-universal beyond the second order. (This owes to the fact that pion fields are “coordinates”, which parameterize the unitary manifold satisfying $U^\dagger U = 1$. We could equally well have chosen, *e.g.*, $U = (1 + i\tilde{\pi})/\sqrt{1 + \pi_a\pi^a/f_\pi^2}$. Upon comparing expansions of both U s, it is evident that universality is lost at $O(\tilde{\pi}^3)$.) Physically, there is no general way to lock the coefficients of higher order terms to lower order ones without additional constraints. Imposing the equations of motion, however, does lock the higher order terms to the universal lower order ones (one must use an expansion in pions in the kinetic term as well as in the WZ term when the equations of motion are implemented). The actual on-mass-shell matrix elements are thus universal. Consequently, the form of the WZ term is universal only at the fifth order in $\text{Tr}(\pi\alpha^4)$, since at the next order we pick up nonuniversal

terms from expansions of α . Moreover, there is no way to insure the self-consistency beyond this order off-mass shell.

V. CONCLUSIONS

We have initiated the discussion as to how the Chern-Simons terms of a $D = 5$ pure Yang-Mills theory can be deformed into the Wess-Zumino-Witten terms of gauged chiral lagrangians of $D = 4$.

Adjoint currents in $D = 5$ are controlled by the second Chern character. This in turn becomes the WZW term in $D = 4$. The minimal coefficient of Witten for the Wess-Zumino term follows from the simplest case of pure gauge vector potentials generated by London currents in the orbifold magnetic superconducting end-zones, as shown in our heuristic argument.

Singlet currents follow from introduction of a singlet $U(1)$ vector potential in $D = 5$ Yang-Mills, which is a dual variable describing the instantonic soliton that uniquely occurs there. We summarize how this morphs into a new WZW term in $D = 4$, involving the σ and η' fields, and which generates the corresponding chiral current equations of motion. A new $U(1)$ axial current, associated with the η' , has also been identified. These results are a consequence of the present approach, and may have application to skyrmion physics.

We then embark upon a formal discussion of the latticization of the extra (fifth) dimension, and study hopping derivatives and the Bianchi identities. The coset Bianchi identity is shown to fail in the case of the diagonal hopping derivative in the fifth dimension, the most natural definition for a lattice gauge theory. We find, however, that the coset Bianchi identity can be rescued if the basic $D = 4$ covariant derivative is modified by the addition of a chiral vector current with the special coefficient of $1/2$.

This result has intriguing implications. For one, it converts the orbifold compactification into an effective periodic compactification. It also provides a Skyrme term in the effective action that must match the topology of the instantonic soliton to the skyrmion. We conjecture that with the fixed coefficient of the Skyrme term provided by the theory, the matching may be quite powerful, leading perhaps to an analytic skyrmion solution and some form of “self-duality.”

We finally examine the WZW term implied by the Bianchi-consistent theory. Again we obtain the WZW term, but now with a coefficient that has an index of $N = D = 5$. Many other issues are raised and future lines to explore are suggested by the present work.

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